Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 4, pp. 651–657, 2009

COMPRESSION OF A PLASTIC POROUS MATERIAL BETWEEN ROTATING PLATES

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UDC 539.374

The problem of a plane flow of a rigid-plastic porous material between two rotating rough plates with no material flux through the point of their rotation and with a uniform distribution of porosity at the initial instant is considered under the assumption that the material behavior follows the cylindrical yield condition and the associated flow rule. The solution reduces to consecutive calculation of several ordinary integrals. It is demonstrated that the solution behavior depends on the angle between the plates, and the value of porosity at a certain stage of the deformation process can be equal to zero.

Key words: *porosity, friction, analytical solution, plasticity.*

In the classical theory of plasticity of incompressible materials, there are many analytical solutions obtained by an inverse method [1, 2]. In the theory of plasticity of porous materials, such solutions seem to be lacking (even for the initial flow), except for a uniform stress-strain state and extremely simple problems with the friction stress neglected, for instance, problems of contraction of a hollow cylinder in a plane strain state (initial flow), contraction of a hollow spherical shell (initial flow), and flow through a convergent channel [3]. The classical problems of the plasticity theory [the most typical problem is the Prandtl problem of compression of a layer between rough plates (see, e.g., [1])], which have fairly simple solutions in the case of an incompressible rigid perfectly/plastic material (and their extensions to other models of incompressible materials [4]), cannot be extended to models of plastically compressible materials. In particular, attempts of problem extension were made in [5] for the Prandtl problem and in [6] for the flow through an infinitely convergent channel. Solutions, however, were obtained in none of these attempts (under the same assumptions at which the classical solutions are constructed). As the exact solutions are of interest and can play an important role in verification of numerical codes [7], the search for such solutions, especially in the presence of friction stresses, is an urgent task.

We assume that the material behavior obeys the cylindrical yield condition proposed in [8] and written in the form

$$\sigma_{\rm eq} \leqslant \sqrt{3}\,\tau_s, \qquad |\sigma| \leqslant p_s. \tag{1}$$

Here $\sigma_{\rm eq}$ is the equivalent stress, σ is the mean stress, τ_s is the shear yield stress, and p_s is the yield stress under hydrostatic compression. The equivalent and mean stresses are defined by the relations $\sigma_{\rm eq} = \sqrt{3/2}\,(\tau_{ij}\tau_{ij})^{1/2}$ and $\sigma = \sigma_{ij}\delta_{ij}/3$, where σ_{ij} are the stress tensor components, τ_{ij} are the stress deviator components, and δ_{ij} is the Kronecker symbol. The quantities p_s and τ_s are assumed to be known functions of porosity η. As $\eta \to 0$, we have $p_s \to \infty$, and quantity τ_s tends to the shear yield stress of the base material. Though the yield condition (1) is extremely simple, it can describe some experimental observations [8]. Other solutions with the use of the yield condition (1) were obtained in [9, 10]. Note that various flow regimes are possible if this condition is applied. Nevertheless, the regime described by the relations $\sigma_{\rm eq} \leqslant \sqrt{3}\,\tau_s$ and $|\sigma| < p_s$ leads to the system of equations of the classical theory of plasticity of incompressible materials, while the regime with $\sigma_{\rm eq} < \sqrt{3}\tau_s$ and $|\sigma| \leqslant p_s$ is

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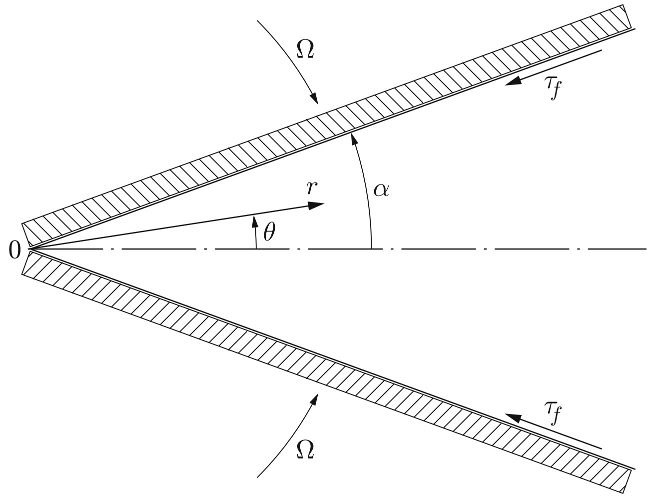

Fig. 1. Geometry of the problem.

impossible in the case of plane deformation. Thus, we assume that there is the sign of equality in Eqs. (1), and $\sigma < 0$. The full system of equations based on the yield condition (1) was given in [11].

Let us consider the compression of a porous plastic material between rough plates rotating with an angular velocity Ω under conditions of a plate strain state (Fig. 1). As the strain rate does not affect the material resistance, we can assume that $\Omega = 1$ without loss of generality. Solutions of the problem considered for some models of plastically incompressible materials were obtained in [12–14]. A particular case of these solutions is the rigid perfectly/plastic solution. Let us introduce a cylindrical coordinate system $r\theta z$ whose z axis coincides with the axis of rotation of the plates. The material occupies the domain $0 \leqslant r < \infty$, $-\alpha \leqslant \theta \leqslant \alpha$ (2α is the apex angle between the plates). There is no material flux through the point $r = 0$; hence, we can assume that the friction stresses τ_f are directed toward this point (see Fig. 1). By virtue of problem symmetry, it is sufficient to obtain the solution at $\theta \geqslant 0$. As $\tau_{rz} = 0$ and $\tau_{\theta z} = 0$, then the first relation in Eqs. (1) is satisfied as a result of the substitution

$$\tau_{r\theta} = -\tau_s \sin\omega, \qquad \tau_{\theta\theta} = -(2/\sqrt{3}\,)\tau_s \cos\omega \sin(\gamma + \pi/3),$$
$$\tau_{rr} = (2/\sqrt{3}\,)\tau_s \cos\omega \cos(\gamma + \pi/6). \tag{2}$$

With allowance for this substitution, the plastic flow rule associated with Eqs. (1) acquires the form

$$\xi_{rr} = 2\lambda_1 \cos\omega \cos(\gamma + \pi/6) - \lambda_2, \qquad \xi_{\theta\theta} = -2\lambda_1 \cos\omega\, \sin(\gamma + \pi/3) - \lambda_2,$$
$$0 = 2\lambda_1 \cos\omega\, \sin\gamma - \lambda_2, \qquad \xi_{r\theta} = -\sqrt{3}\,\lambda_1 \sin\omega, \qquad \lambda_1 \geqslant 0, \quad \lambda_2 \geqslant 0. \tag{3}$$

Eliminating the terms λ_1 and λ_2 from Eqs. (3), we obtain

$$\frac{\xi_{rr} - \xi_{\theta\theta}}{\xi_{r\theta}} = -2\cot\,\omega \cos\gamma, \qquad \frac{\xi_{rr} + \xi_{\theta\theta}}{\xi_{r\theta}} = 2\sqrt{3}\,\cot\,\omega \sin\gamma. \tag{4}$$

The inequalities in Eqs. (3) are satisfied if

$$\cos\omega \sin\gamma > 0, \qquad \sin\omega > 0. \tag{5}$$

The assumption made in (5) that

$$\xi_{r\theta} < 0 \tag{6}$$

follows from the flow character (see Fig. 1) and has to be verified after the solution is obtained.

Let the stress deviator components be independent of r. As the porosity is uniformly distributed, then we have $\tau_s = \text{const}$ and $p_s = \text{const}$. Thus, it follows from Eqs. (1) that $\sigma = \text{const}$, and the equilibrium equations acquire the form

$$\frac{d\tau_{r\theta}}{d\theta} + \tau_{rr} - \tau_{\theta\theta} = 0, \qquad \frac{d\tau_{\theta\theta}}{d\theta} + 2\tau_{r\theta} = 0. \tag{7}$$

Substituting relations (2) into Eqs. (7), we obtain

$$\frac{d\omega}{d\theta} = 2\cos\gamma, \qquad \sin\Big(\gamma + \frac{\pi}{3}\Big)\frac{d\omega}{d\theta} - \cot\omega\,\cos\Big(\gamma + \frac{\pi}{3}\Big)\frac{d\gamma}{d\theta} = \sqrt{3}. \tag{8}$$

Passing in the second equation of (8) to differentiation with respect to ω and eliminating the derivative $d\omega/d\theta$ with the help of the first equation, we find

$$\frac{d\gamma}{d\omega} = \tan\omega\,\tan\gamma. \tag{9}$$

The general solution of Eq. (9) has the form

$$\cos\omega = C/\sin\gamma. \tag{10}$$

It follows from Eq. (10) and inequalities (5) that $1 > C > 0$. Taking into account the character of the flow (see Fig. 1), we can assume that

$$\xi_{rr} - \xi_{\theta\theta} > 0. \tag{11}$$

[Inequality (11) should be verified after the solution is constructed.] Then, taking into account Eq. (5), we can assume without loss of generality that

$$\cos\omega > 0, \quad \cos\gamma > 0, \quad \sin\gamma > 0, \qquad 0 < \omega < \pi/2, \quad 0 < \gamma < \pi/2. \tag{12}$$

If inequalities (12) are satisfied, then the substitution of Eq. (10) into the first equation of system (8) yields the relation

$$\frac{d\omega}{d\theta} = 2\Big(1 - \frac{C^2}{\cos^2\omega}\Big)^{1/2}. \tag{13}$$

The general solution of Eq. (13) has the form

$$\arctan\,(\sqrt{2}\,\sin\omega/(1 - 2C^2 + \cos 2\omega)^{1/2}) = 2\theta + C_1. \tag{14}$$

As $\tau_{r\theta} = 0$ on the axis of symmetry, the dependence $\omega(\theta)$ is found in an implicit form from relations (2) and (14):

$$\sqrt{2}\sin\omega/(1 - 2C^2 + \cos 2\omega)^{1/2} = \tan\,2\theta. \tag{15}$$

The dependence $\gamma(\theta)$ is found from Eqs. (10) and (15) in a parametric form.

Let us present the velocity field as

$$u_r = rU_r(\theta), \qquad u_\theta = rU_\theta(\theta), \tag{16}$$

where the quantities $U_r(\theta)$ and $U_\theta(\theta)$ depend only on θ. In such a form, the velocity vector satisfies the condition of the absence of material flux through the point $r = 0$ with an arbitrary choice of the functions $U_r(\theta)$ and $U_\theta(\theta)$. The boundary conditions $u_\theta = 0$ at $\theta = 0$ and $u_\theta = -r$ at $\theta = \alpha$ yield the conditions for U_θ:

$$U_\theta = 0 \qquad \text{at} \quad \theta = 0; \tag{17}$$

$$U_\theta = -1 \qquad \text{at} \quad \theta = \alpha. \tag{18}$$

Calculating the strain rate tensor components from equalities (16) and substituting them into Eq. (4), we obtain

$$\frac{dU_\theta}{d\theta} = \cot\omega\,\cos\gamma\,\frac{dU_r}{d\theta}, \qquad \frac{dU_\theta}{d\theta} + 2U_r = \sqrt{3}\,\cot\omega\,\sin\gamma\,\frac{dU_r}{d\theta}. \tag{19}$$

Eliminating the derivative $dU_\theta/d\theta$ from the second equation in (19) with the help of the first equation and passing in the resultant expression to differentiation with respect to γ with the help of Eqs. (8) and (9), we find

$$\frac{dU_r}{U_r} = \frac{d\gamma}{\sin\gamma\,(\sqrt{3}\,\sin\gamma - \cos\gamma)}. \tag{20}$$

The general solution of Eq. (20) has the form

$$U_r = C_2(\sqrt{3}\sin\gamma - \cos\gamma)/\sin\gamma. \tag{21}$$

Passing to differentiation with respect to γ in the first equation of (19), eliminating U_r with the help of Eq. (21), and taking into account Eqs. (5) and (10), we obtain

$$\frac{dU_\theta}{d\gamma} = \frac{CC_2 \cos\gamma}{\sin^2\gamma(\sin^2\gamma - C^2)^{1/2}}. \tag{22}$$

For determining the constants of integration, we have to specify the law of friction at $\theta = \alpha$. Let us use the maximum friction law, which implies that the friction stresses reach the maximum possible values in the case of slipping [15], and singular velocity fields are formed if certain models of the material are used [13–15]. Moreover, the solution does not exist at higher values of the shear stress on the contact surface. For the material model considered, we find the following relations from Eq. (2) in the case of slipping:

$$\omega = \pi/2 \qquad \text{at} \quad \theta = \alpha. \tag{23}$$

First, we have to construct the solution with allowance for the no-slip condition on the surface $\theta = \alpha$: $u_r|_{\theta=\alpha} = 0$. Then, Eqs. (16) and (21) yield

$$\gamma = \pi/6 \qquad \text{at} \quad \theta = \alpha. \tag{24}$$

We assume that $\omega = \omega_s$ at $\theta = \alpha$ and $\gamma = \gamma_0$ at $\theta = 0$. Then, taking into account that $\omega = 0$ at $\theta = 0$, we find the following relations from Eqs. (10), (12), (15), and (24):

$$C = \sin\gamma_0, \qquad \cos\omega_s = 2\sin\gamma_0, \qquad \tan\,\omega_s = (\sqrt{3}/2)\tan\,2\alpha. \tag{25}$$

Taking into account equalities (17) and (25), we write the solution of Eq. (22) in the form

$$U_\theta = \sin\gamma_0 C_2 \int\limits_{\gamma_0}^{\gamma} \frac{\cos y}{\sin^2 y(\sin^2 y - \sin^2\gamma_0)^{1/2}}\, dy. \tag{26}$$

The constant C_2 is determined from equalities (18) and (26) as follows:

$$C_2 = -\Big(\sin\gamma_0 \int\limits_{\gamma_0}^{\gamma_s} \frac{\cos\gamma}{\sin^2\gamma(\sin^2\gamma - \sin^2\gamma_0)^{1/2}}\, d\gamma\Big)^{-1}. \tag{27}$$

Thus, for the chosen value of α, the dependences of ω, U_r, and U_θ on the parameter γ are determined from Eqs. (10), (21), (26), and (27). Then, the distributions of stresses and velocities can be found from Eqs. (2) and (16). In addition, using relation (15), we can find the dependence of the sought functions on θ in a parametric form. Note that the integral in Eq. (26) is not a proper integral, but its convergence can be easily demonstrated. It follows from Eqs. (25) that the solution obtained is only valid at $0 < \alpha \leqslant \pi/4$. The case $\alpha = \pi/4$ has to be considered separately.

It also follows from Eqs. (25) that condition (23) is satisfied at $\alpha = \pi/4$; in this case, we have $C = 0$. According to Eqs. (3) and (10), we have $\lambda_2 = 0$; therefore, the flow regime is changed: the strain rate vector (in the space of the principal strain rates superposed with the space of the principal stresses) becomes orthogonal to the side surface of the cylinder defined by Eqs. (1). In particular, the condition of incompressibility holds. The solution for this case is obtained at $\gamma = 0$. In Eqs. (20) and (24), however, it is necessary to eliminate differentiation with respect to γ, which is more convenient to do with the use of Eqs. (19). As Eq. (15) with $C = 0$ yields $\omega = 2\theta$, then Eqs. (19) can be transformed as

$$\frac{dU_\theta}{d\theta} = \cot\,(2\theta)\,\frac{dU_r}{d\theta}, \qquad \frac{dU_\theta}{d\theta} + 2U_r = 0. \tag{28}$$

The solution of Eqs. (28) satisfying conditions (17) and (18) has the form

$$U_r = \cos 2\theta, \qquad U_\theta = -\sin 2\theta. \tag{29}$$

Though the friction law (23) does not prohibit the slipping regime in this case, it follows from solution (29) that $U_r = 0$ at $\theta = \alpha$. Such solutions can exist in cases with particular geometric properties of the friction surface [15]. Note that the solution at $\alpha = \pi/4$ was constructed with the use of two equations of the yield condition (1), though one of then does not affect the magnitude of the strain rate tensor components. In particular, the second equation of system (28) is the condition of incompressibility. At $\alpha > \pi/4$, the solution should be constructed under the

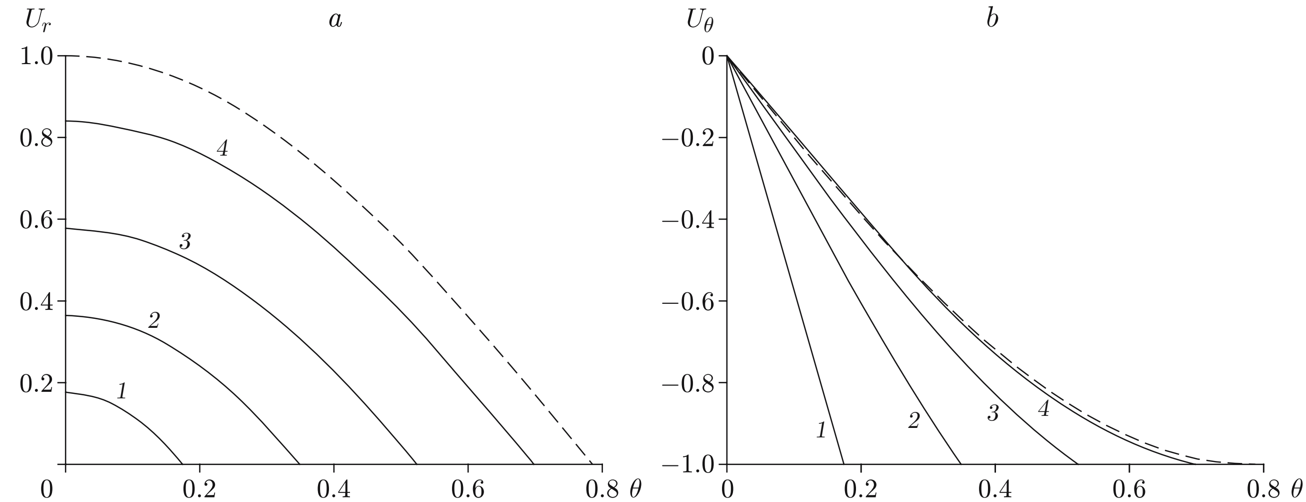

Fig. 2. Dependences $U_r(\theta)$ (a) and $U_\theta(\theta)$ (b) for $\alpha = \pi/18$ (1), $\pi/9$ (2), $\pi/6$ (3), and $2\pi/9$ (4); the dashed curves show the solution obtained from Eq. (29) at $\alpha = \pi/4$.

assumption that only the equation $\sigma_{\rm eq} = \sqrt{3}\,\tau_s$ holds. As a result, we obtain the system of equations of the classical theory of plasticity of incompressible materials; therefore, the corresponding solution is not presented here. In this case, the no-slip condition is also satisfied. Thus, the no-slip regime on the friction surface is obtained for the material model and boundary-value problem considered if the maximum friction law is used. This behavior of the solutions is typical for some other models of rigid-plastic materials [14].

The distribution of the stress tensor components in the circumferential direction is determined in an analytical form from Eqs. (1), (2), (10), (15), and (25). Figure 2 shows the dependences $U_r(\theta)$ and $U_\theta(\theta)$ for various values of the angle α, which were obtained from Eq. (21) and by applying numerical integration in Eqs. (26) and (27).

If the velocity fields are described by relations (16), inequalities (6) and (11) acquire the form $dU_r/d\theta < 0$ and $dU_\theta/d\theta < 0$. It is seen from Fig. 2 that these inequalities are satisfied.

Under the assumption that the value of porosity is arbitrary and its distribution is uniform, the resultant solution describes the instantaneous stress-strain state. This solution, however, can be extended to the case of deformation of a material with a varied porosity. It follows from the relation for the velocity field (16) that the first invariant of the strain rate tensor is determined by the relation $\xi = \xi_{rr} + \xi_{\theta\theta} = 2U_r + dU_\theta/d\theta$, which acquires the following form with allowance for Eqs. (8), (9), and (19)–(21):

$$\xi = 2\sqrt{3}\,C_2. \qquad (30)$$

As the quantity C_2 during the deformation process is a function of the time t (or the angle α), the continuity equation is written in the form

$$\frac{d(1-\eta)}{dt} = -(1-\eta)\xi, \qquad (31)$$

where d/dt is the total derivative with respect to time. The assumption about the coordinate-independent porosity does not contradict the initial condition $\eta = \eta_0 = \text{const}$ at $t = 0$ (or $\alpha = \alpha_0$) and also Eqs. (30) and (31); hence, the total derivative with respect to time in Eq. (31) can be replaced by the local derivative. Eliminating the parameter ξ from Eq. (31) with the use of Eq. (30), passing to differentiation with respect to α with the help of the relation $d\alpha/dt = -1$, and integrating the resultant expression under the initial condition $\eta = \eta_0$ ($\alpha = \alpha_0$), we obtain

$$\eta = 1 - (1-\eta_0)\exp\Big(2\sqrt{3}\int\limits_{\alpha_0}^{\alpha} C_2\,d\alpha\Big). \qquad (32)$$

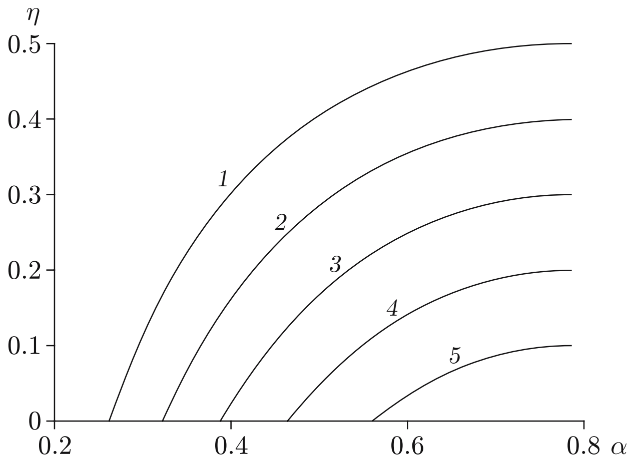

Fig. 3

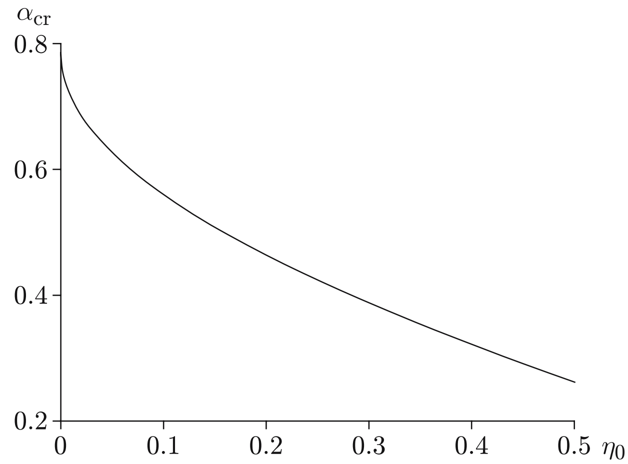

Fig. 4

Fig. 3. Porosity versus the angle between the plates at $\alpha_0 = \pi/4$ and different values of the initial porosity: $\eta_0 = 0.5$ (1), 0.4 (2), 0.3 (3), 0.2 (4), and 0.1 (5).

Fig. 4. Angle between the plates at which the porosity becomes equal to zero versus the initial porosity ($\alpha_0 = \pi/4$).

Assuming that $\eta = 0$ in Eq. (32), we obtain the following relation for determining the angle $\alpha = \alpha_{\rm cr}$ at which the material reaches a pore-free state:

$$1 = (1 - \eta_0) \exp\Big(2\sqrt{3} \int\limits_{\alpha_0}^{\alpha_{\rm cr}} C_2 \, d\alpha\Big). \tag{33}$$

As the dependence $C_2(\alpha)$ is determined by Eqs. (25) and (27), then the integration in Eqs. (32) and (33) can be performed numerically. Figure 3 shows the behavior of porosity in the course of deformation for different values of the initial porosity. The angle $\alpha_{\rm cr}$ as a function of the initial porosity is plotted in Fig. 4.

Thus, an analytical solution of the initial-boundary-value problem of the flow of a rigid-plastic porous material between two rotating rough plates is obtained. It is demonstrated that the no-slip regime is observed on the friction surface if the maximum friction law is used. Various flow regimes occur on the yield surface, depending on the angle between the plates; in particular, the flow of the material with no changes in porosity is possible.

This work was supported by the Russian Foundation for Basic Research (Grant No. 08-01-92011_NNS_a) and by the Council on the Grants of the President of the Russian Federation for Supporting the Leading Scientific Schools (Grant No. NSh-134.2008.1).

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